2010 年度日本政府(文部科学省)奨学金留学生選考試験

QUALIFYING EXAMINATION FOR APPLICANTS FOR JAPANESE

GOVERNMENT (MONBUKAGAKUSHO) SCHOLARSHIPS 2010

学科試験　　問題

EXAMINATION QUESTIONS

(学部留学生)

UNDERGRADUATE STUDENTS

数　　学（B）

MATHEMATICS(B)

注意　☆試験時間は 60 分。

PLEASE NOTE : THE TEST PERIOD IS 60 MINUTES

Q 1 The quadratic function which takes the value 41 at $x = -2$ and the value 20 at $x = 5$ and is minimized at $x = 2$ is

$$y = \boxed{\mathsf{A}}\,x^2 - \boxed{\mathsf{B}}\,x + \boxed{\mathsf{C}}.$$

The minimum value of this function is $\boxed{\mathsf{D}}$.

Q 2 Consider the integral expression in x

$$P = x^3 + x^2 + ax + 1,$$

where a is a rational number.

At $a =$ **A** the value of P is a rational number for any x which satisfies the equation $x^2 + 2x - 2 = 0$, and in this case the value of P is **B**.

Q 3 For each of **A** ~ **D** in the following statements, choose the most appropriate expression from among ⓪~⑨ below.

Consider two conditions $x^2 - 3x - 10 < 0$ and $|x - 2| < a$ on a real number x, where a is a positive real number.

(i) A necessary and sufficient condition for $x^2 - 3x - 10 < 0$ is that **A** $< x <$ **B** .

(ii) The range of values of a such that $|x - 2| < a$ is a necessary condition for $x^2 - 3x - 10 < 0$ is **C** .

(iii) The range of values of a such that $|x - 2| < a$ is a sufficient condition for $x^2 - 3x - 10 < 0$ is **D** .

⓪ 2	① 5	② -2	③ -5
④ $a \geqq 2$	⑤ $a \geqq 3$	⑥ $a \geqq 4$	
⑦ $0 < a \leqq 2$	⑧ $0 < a \leqq 3$	⑨ $0 < a \leqq 5$	

Q 4 Let d be the common difference of an arithmetic progression $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ which satisfies the two conditions

$$a_5 a_7 - a_4 a_9 = 60, \qquad a_{11} = 25.$$

Then

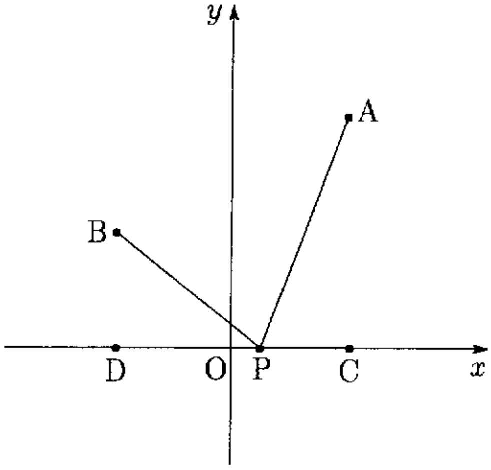
(1) Either $d =$ **A** or $d =$ **B**, where A > B.

(2) If $d =$ A, then $a_1 =$ **C**, $a_n =$ **D** $n -$ **E**, and the sum of the first n terms is 195 when $n =$ **F**.

Q 5 Consider the four points

A(1,2), B(−1,1), C(1,0), D(−1,0)

on the xy-plane. Let P be a point on the segment CD, excluding the endpoints. We are to find the point P at which ∠APB is maximized.



Set $\alpha = \angle\mathrm{APC}$, $\beta = \angle\mathrm{BPD}$ and $\theta = \angle\mathrm{APB}$, and let t be the x-coordinate of point P. Then

$$\tan\alpha = \frac{\boxed{\mathsf{A}}}{\boxed{\mathsf{B}} - t}, \quad \tan\beta = \frac{\boxed{\mathsf{C}}}{\boxed{\mathsf{D}} + t}$$

and hence

$$\tan\theta = \frac{t + \boxed{\mathsf{E}}}{t^2 + \boxed{\mathsf{F}}}.$$

When we differentiate the right-hand side of this equation with respect to t, we have

$$\left(\frac{t + \boxed{\mathsf{E}}}{t^2 + \boxed{\mathsf{F}}}\right)' = -\frac{t^2 + \boxed{\mathsf{G}}\,t - 1}{(t^2 + \boxed{\mathsf{H}})^2}.$$

Therefore the coordinates of point P are

$$\left(\boxed{\mathsf{I}},\ 0\right).$$

Q 6 Let α be a real number. Let us translate the graph of the cubic function

$$y = f(x) = x^3 + ax^2 + bx + c \qquad \cdots\cdots\cdots \quad ①$$

so that the point $(\alpha, f(\alpha))$ on the graph of ① is translated into the origin $(0,0)$, and express the function of the translated graph in terms of $f'(\alpha)$ and $f''(\alpha)$.

First, we have

$$f'(\alpha) = \boxed{\textsf{A}}\,\alpha^2 + \boxed{\textsf{B}}\,a\alpha + b \qquad \cdots\cdots\cdots \quad ②$$

$$f''(\alpha) = \boxed{\textsf{C}}\,\alpha + \boxed{\textsf{D}}\,a. \qquad \cdots\cdots\cdots \quad ③$$

Next, we consider the translation which translates the point $(\alpha, f(\alpha))$ on the graph of ① into the origin, that is, we replace x with $x+\alpha$ and y with $y+f(\alpha)$ in ①, and obtain the expression

$$y = x^3 + \frac{f''(\alpha)}{\boxed{\textsf{E}}}\,x^2 + f'(\alpha)x$$

by using ② and ③.

As an example, consider the function

$$f(x) = x^3 - 12x^2 + 48x - 68. \qquad \cdots\cdots\cdots \quad ④$$

As

$$f'(\boxed{\textsf{F}}) = 0 \quad \text{and} \quad f''(\boxed{\textsf{G}}) = 0,$$

we see that when we translate the graph of ④ so that the point $(\boxed{\textsf{H}}, \boxed{\textsf{I}})$ on the graph is moved to the origin, we get the graph of $y = x^3$.

Q 7 Consider the curve $y = 2\log x$, where log is the natural logarithm. Let ℓ be the tangent to that curve which passes through the origin, let P be the point of contact of ℓ and that curve, and let m be the straight line perpendicular to the tangent ℓ at P. We are to find the equations of the straight lines ℓ and m and the area S of the region bounded by the curve $y = 2\log x$, the straight line m, and the x-axis.

Let t be the x-coordinate of the point P. Then t satisfies $\log t = \boxed{\text{A}}$. Hence the equation of ℓ is

$$y = \frac{\boxed{\text{B}}}{e}x.$$

The equation of m is

$$y = -\frac{e}{\boxed{\text{C}}}x + \frac{e^2}{\boxed{\text{D}}} + \boxed{\text{E}}.$$

Thus the area S of the region is

$$S = \boxed{\text{F}} + \frac{\boxed{\text{G}}}{e}.$$